

# SCHRÖDINGER EQUATION WITH LINEAR POTENTIAL AND HITTING TIMES

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**ABSTRACT.** In Hernández-del-Valle (2010) the author studies the connection between Schrödinger's equation and first hitting densities of Brownian motion. Although the author is able to find solutions of a Schrödinger type pde he fails—except in some special cases—to construct a solution which satisfies the boundary on the space variable at  $x = 0$ . In this paper we use an approach used in Bluman and Shtelen (1996) to find solutions which satisfy the pde and boundary condition when  $t = 0$ .

Consider the Schrödinger equation

$$(1) \quad \frac{\partial u_1}{\partial t}(t, x) + \frac{\partial^2 u_1}{\partial x^2}(t, x) - V_1(t, x)u_1(t, x) = 0.$$

Given a linear operator  $\mathcal{L}$ , its adjoint  $\mathcal{L}^*$  is defined by

$$\Phi \mathcal{L}u - u \mathcal{L}^* \Phi = \sum_{i=1}^n D_i f^i$$

where  $x = (x_1, x_2, \dots, x_n)$ , the total derivative operators  $D_i = \partial/\partial x_i$ ,  $i = 1, 2, \dots, n$ , and  $\{f^i\}$  are bilinear expression in  $u, \Phi$  and their derivatives. Consequently, if

$$(2) \quad \mathcal{L}^* \Phi = 0$$

then  $\mathcal{L}u = 0$  if and only if  $\sum_{i=1}^n D_i f^i = 0$ , i.e. a given linear partial differential equation

$$(3) \quad \mathcal{L}u = 0$$

is equivalent to the conservation law

$$(4) \quad \sum_{i=1}^n D_i f^i = 0$$

for any  $\Phi$  satisfying its adjoint equation (2).

We now specialize to the case when (3) is the Schrödinger equation (1). Here the linear operator is

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - V_1(t, x)$$

its adjoint is given by

$$\mathcal{L}^* = -\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - V_1(t, x)$$

and (4) becomes

$$(5) \quad \frac{\partial}{\partial t}(\Phi u_1) + \frac{\partial}{\partial x} \left( \Phi \frac{\partial u_1}{\partial x} - \frac{\partial \Phi}{\partial x} u_1 \right) = 0.$$

The potential system corresponding to (5) is given by

$$(6) \quad \frac{\partial v_1}{\partial x} = \Phi u_1$$

$$(7) \quad \frac{\partial v_1}{\partial t} = \frac{\partial \Phi}{\partial x} u_1 - \Phi \frac{\partial u_1}{\partial x}$$

where  $\Phi(t, x)$  is a solution of

$$(8) \quad \mathcal{L}^* \Phi = -\frac{\partial \Phi}{\partial t} + \frac{\partial^2 \Phi}{\partial x^2} - V_1(t, x) \Phi = 0.$$

Note that if  $(u_1(t, x), v_1(t, x), \Phi(t, x))$  solves (6), (7) and (8) then  $u_1(t, x)$  solves Schrödinger equation (1) and  $v_1(t, x)$  solves

$$(9) \quad \frac{\partial v_1}{\partial t} + \frac{\partial^2 v_1}{\partial x^2} - \frac{2}{\Phi} \frac{\partial \Phi}{\partial x} \frac{\partial v_1}{\partial x} = 0.$$

If  $u_1(t, x)$  solves (1) and  $\Phi(t, x)$  solves (8), then one can  $v_1(t, x)$  solving (6) and (7) i.e. for any  $\Phi(t, x)$  satisfying (8) the point transformation

$$(10) \quad w = \frac{v_1}{\Phi}$$

maps (9) to

$$(11) \quad \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} - V_2(t, x)w = 0$$

where the new potential  $V_2(t, x)$  is given by

$$(12) \quad V_2(t, x) = V_1(t, x) - 2 \frac{\partial^2}{\partial x^2} \log \Phi.$$

From equations (6), (7) and (10) it follows that if  $u_1(t, x)$  solves (1) then

$$(13) \quad w(t, x) = \frac{1}{\Phi(t, x)} \left[ \int_k^x u_1(t, \xi) \Phi(t, \xi) d\xi + B_2(t) \right]$$

with  $B_2(t)$  satisfying the condition

$$(14) \quad \frac{dB_2}{dt} = \frac{\partial \Phi}{\partial x}(k, t) u_1(t, k) - \Phi(t, k) \frac{\partial u_1}{\partial x}(t, k)$$

for any constant  $k$ , solves the Schrödinger equation (10).

In particular, let  $V_1(t, x) = xf''(t)$  then (8) is

$$\frac{\partial \Phi}{\partial t}(t, x) + xf''(t)\Phi(t, x) = \frac{\partial^2 \Phi}{\partial x^2}(t, x).$$

which in turn admits solutions of the following form:

$$\Phi(t, x) = \exp \left\{ \frac{1}{2} \int_0^t (f'(u))^2 du - xf'(t) \right\} \omega \left( t, x - \int_0^t f'(s) ds \right)$$

where  $\omega$  in turn is a solution of

$$(15) \quad \omega_t = \omega_{xx}.$$

a particular solution to (15) is given by

$$\omega(t, x) = \exp \left\{ -\frac{1}{2} \lambda^2 t \pm \lambda x \right\}$$

for some scalar  $\lambda$ , which alternatively implies that a solution to (10) is given by:

$$(16) \quad \Phi(t, x) = e^{\frac{1}{2} \int_0^t (f'(u))^2 du - x[f'(t) \pm \lambda] - \frac{1}{2} \lambda^2 t \pm \int_0^t f'(u) du}.$$

Next, note that the potential  $V_2(t, x) = V_1(t, x)$  since

$$\frac{\partial^2}{\partial x^2} \log \Phi = 0.$$

Hence (11) becomes:

$$(17) \quad -\frac{\partial w}{\partial t}(t, x) + xf''(t)w(t, x) = \frac{1}{2} \frac{\partial^2 w}{\partial x^2}(t, x)$$

or from (13) and setting  $k = 0$

$$w(t, x) = \frac{1}{\Phi(t, x)} \left[ \int_0^x u_1(t, \xi) \Phi(t, \xi) d\xi + B_2(t) \right],$$

where  $u_1$  solves Schrödinger's equation (1),  $\Phi$  is as in (16) and  $B_2$  is defined in (14).

Solutions of  $u_1$  are:

$$u_1(t, x) = \exp \left\{ \frac{1}{2} \int_t^s (f'(u))^2 du + xf'(t) \right\} \omega \left( s - t, x + \int_t^s f'(u) du \right).$$

Then  $w(0, x)$  becomes:

$$w(0, x) = \frac{1}{\Phi(0, x)} \int_0^x u_1(0, \xi) \Phi(0, \xi) d\xi$$

and as  $x \rightarrow 0$  then  $w \rightarrow 0$ .

The problem is now how to choose  $\omega$ .

From Proposition 2.3 and Proposition 3.1 in Hernández-del-Valle we have

$$\begin{aligned} -\frac{\partial v}{\partial t} + x f''(t)v &= \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \left( \frac{1}{x} - \frac{x}{s-t} \right) \frac{\partial v}{\partial x} \\ v(s, x) &= 1 \end{aligned}$$

and

$$(18) \quad 0 \leq v(t, x) \leq 1$$

admits:

$$v(t, x) := \tilde{\mathbb{E}}^{t,x} \left[ \exp \left\{ - \int_t^s f''(u) \tilde{X}_u du \right\} \right].$$

and given that

$$h(t, x) = \frac{x}{\sqrt{2\pi t^3}} \exp \left\{ -\frac{x^2}{2t} \right\}$$

then

$$v(t, x) = \frac{w(t, x)}{h(s-t, x)}$$

where  $w$  is an (17). From (18)

$$0 \leq w(t, x) \leq h(s-t, x)$$

and hence as  $x \rightarrow 0$  then  $w \rightarrow 0$ . This will happen for all  $t$  if  $B_2 = 0$ . Else it holds for  $t = 0$ .

## REFERENCES

- [1] Bluman, G. and V. Shtelen (1996). New Classes of Schrödinger equations equivalent to the free particle equation through non-local transformations. *J. Phys. A: Math Gen.* **29** 4473–4480.
- [2] Hernández-del-Valle, G. (2010). On Schrödinger's equation, 3-dimensional Bessel Bridges, and Passage Time Problems. *submitted*.